Writing Game Theory in LaTeX

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List of Figures and Tables

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In order to present how to write game theory in LaTeX, I will use the most popular game in game theory called the Prisoner's Dilemma (PD) as an example. The PD shows why two rational individuals might not cooperate even if cooperation is in their best interest, thus resulting in a sub-optimal outcome. The game was formalized and named by Albert Tucker.[1]

The Prisoner's Dilemma (in static or normal-form game) consist of:

- **A set of players** N, and $N = \{1, 2\}$, where 1 stands for "Player 1" and 2 stands for "Player 2."

- For each $i \in N$, a set of actions S_i—that is, **a set of actions** or a set of strategies—$S_i = \{C, D\}$, where C stands for "Cooperate" and D stands for "Defect." In PD $S_1 = S_2 = \{C, D\}$.

- For each $i \in N$, a **preference relation** $\succsim_i$ over $S_1 \times S_2$ (i.e., the Cartesian product). Instead of outcome, we usually use the term *action profiles* (or strategy profiles), i.e., a combination of actions such as (DC), (CC), and so on. Players care about their actions, because their actions lead to action profiles that have payoff utilities assigned to them. Each player has a utility function $v_i : S_1 \times S_2 \rightarrow \mathbb{R}$. For any collection of sets $S_1, S_2, \ldots, S_n$, we define $S_1 \times S_2 \times \cdots \times S_n = \{(s_1, s_2, \ldots, s_n) | s_1 \in S_1, \ldots, s_n \in S_n\}$. We call $(s_1, s_2, \ldots, s_n)$ the ordered "n-tuples" ordered pairs (s_1, s_2). This is important to show that $(1, 2) \neq (2, 1)$. For PD game $S_1 \times S_2 = \{CC, CD, DC, DD\}$.

The ordering of the action profiles, from best to worst—where the first action in parentheses represents player 1's action and the second action represents player 2's action—is (D, C), where player 1 defects and player 2 cooperates; (C, C), where both 1 and 2 cooperate; (D, D), where both 1 and 2 defect, and; (C, D), where 1 cooperates and 2 defects.

The players' preferences can be represented according to payoff functions. First, we assign a utility function u, such as u_1 for player 1, and u_2 for player 2. We can express the order of

[1]See Poundstone, William. 1992 *Prisoner's Dilemma.* New York City: Anchor Books.

players' preferences as,

$$\begin{aligned}&\text{For player 1:}\\&u_1(D,C) > u_1(C,C) > u_1(D,D) > u_1(C,D)\\&\text{For player 2:}\\&u_2(C,D) > u_2(C,C) > u_2(D,D) > u_1(D,C)\end{aligned} \tag{1}$$

Then, we can assign the respective payoffs,

$$\begin{aligned}&\text{For } P_1\text{:}\\&u_1(D,C) = 3 > u_1(C,C) = 2 > u_1(D,D) = 1 > u_1(C,D) = 0\\&\text{For } P_2\text{:}\\&u_2(C,D) = 3 > u_2(C,C) = 2 > u_2(D,D) = 1 > u_2(D,C) = 0\end{aligned} \tag{2}$$

We can represent the game in a payoff matrix, also called "normal-form game":

Table 1: 2x2 Matrix: Prisoner's Dilemma Normal-Form Game

		Player 2	
		C	D
Player 1	C	$2, 2$	$0, 3$
	D	$3, 0$	$1, 1$

The traditional Prisoners's Dilemma can be generalized from its original setting (see the right matrix below): If both players cooperate, they both receive the reward payoff R for cooperating. If both players defect, they both receive the punishment payoff P. If 1 defects while 2 cooperates, then 1 receives the temptation payoff T, while 2 receives the "sucker's" payoff, S. Symmetrically, if 1 cooperates while 2 defects, then 1 receives the sucker's payoff S, while 2 receives the temptation payoff T.

Figure 2: Two Matrices: Generalization of Prisoner's Dilemma

		P_2	
		Cooperate	Defect
P_1	Cooperate	$2, 2$	$0, 3$
	Defect	$3, 0$	$1, 1$

		P_2	
		Cooperate	Defect
P_1	Cooperate	R, R	S, T
	Defect	T, S	P, P

Finding the Nash equilibrium (NE) of the Prisoner's Dilemma (PD) game: The action profile a^* in a strategic game with ordinal preferences is a Nash equilibrium if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a^*_{-i}) in which player i chooses a_i while every other player $-i$, with $i \neq -i$, chooses a^*_{-i}. Therefore, for every player i

$$u_i(a^*) \geq u_i(a_i, a^*_{-i}) \, \forall \, a_i \in A_i \tag{3}$$

where A_i is the set of actions for player i, and u_i is a payoff function that represents player i's preferences.

Table 3: Finding the Nash Equilibrium of Prisoner's Dilemma (Underlining Players' Best Responses)

		Player 2	
		C	D
Player 1	C	$2, 2$	$0, \underline{3}$
	D	$\underline{3}, 0$	$\underline{1}, \underline{1}$

Each underlined action indicates the best response of each player according to the other player's action. Therefore, $NE = (D, D)$.

The PD can also be depicted in a game of extensive-form (or decision tree), such as:

Figure 4: Prisoner's Dilemma in Extensive-form

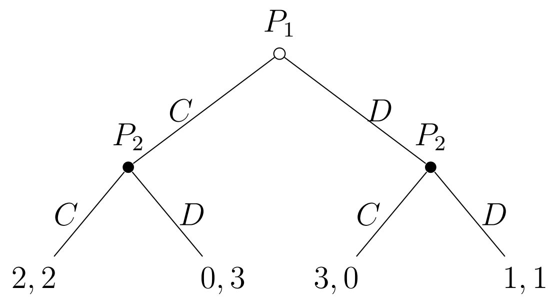
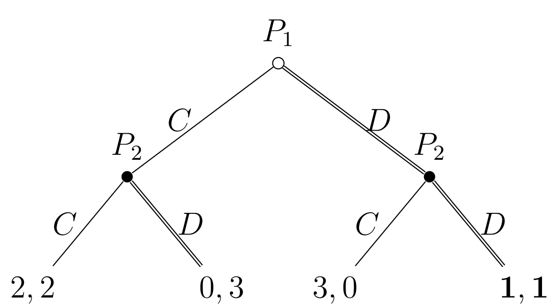


Figure 5: Finding the Subgame Perfect Nash Equilibrium in Prisoner's Dilemma (Using Double Lines)



Other Examples and Game Forms

Table 6: Finding the Nash Equilibrium (Using Dots)

		Player 2 C	Player 2 D
Player 1	C	$2, 2$	$0, \dot{3}$
	D	$\dot{3}, 0$	$\dot{1}, \dot{1}$

The small dot on top of each payoff indicates the best response of each player according to the other player's action. **Note:** Be careful to not confuse strategy profiles (or outputs) with payoff utilities (i.e., $(D, D) \neq (1, 1)$). The equilibrium (or equilibria) of a game refers to the strategy profile(s). Therefore, $NE = (D, D)$.

An alternative depiction of players' strategy profiles and their respective payoff utilities (e.g., PD in normal-form game):

$$\text{For all } \mathrm{a}_i \in A_i, \nu_i(a_i, a_{-i}) = \begin{cases} 3 & \text{if} \quad (D, C) \\ 2 & \text{if} \quad (C, C) \\ 1 & \text{if} \quad (D, D) \\ 0 & \text{if} \quad (C, D) \end{cases}$$

Table 7: 3x3 Matrix: A Game with Three Actions

		Player 2 Cooperate	Player 2 Defect	Player 2 Neither
Player 1	Cooperate	R, R	S, T	T, S
	Defect	T, S	P, P	R, S
	Neither	T, S	P, P	S, S

Table 8: 2x4 Matrix: A Game with Two Actions for P_1 and Four Actions for P_2

		P_2: C Unconditionally	D Unconditionally	Imitation Move	Opposite Move
P_1	C	R, R	S, T	R, R	S, T
	D	T, S	P, P	P, P	T, S

Table 9: Bach or Stravinsky?

		Player 2: Bach	Stravinsky
Player 1	Bach	3, 2	0, 0
	Stravinsky	0, 0	2, 3

Figure 10: The Game of Chicken (The Hawk-Dove Game)

		P_2: Swerve	Straight
P_1	Swerve	0, 0	−1, 1
	Straight	1, −1	−10, −10

		P_2: Swerve	Straight
P_1	Swerve	Tie, Tie	Lose, Win
	Straight	Win, Lose	Crash, Crash

Table 11: Matching Pennies

		Player 2: Heads	Tails
Player 1	Heads	1, −1	−1, 1
	Tails	−1, 1	1, −1

Table 12: Game with Mixed Strategies (Example 1)

		P_2: (q) A	$(1-q)$ B
P_1	(p) A	α, β	ϵ, ζ
	$(1-p)$ B	γ, δ	η, θ

Table 13: Game with Mixed Strategies (Example 2)

		P_2	
		(q) D	$(1-q)$ E
P_1	(x) A	ι, κ	o, π
	(y) B	λ, μ	ρ, σ
	$(1-x-y)$ C	ν, ξ	τ, υ

Figure 14: Centipede Game

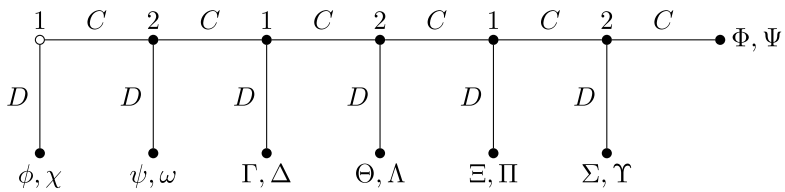


Figure 15: Three Players Game: Combining Extensive Form with Matrix Form

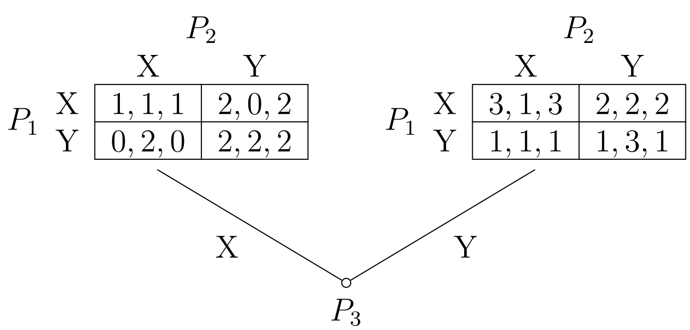


Figure 16: Alternative Three Players Game: Combining Extensive Form with Matrix Form

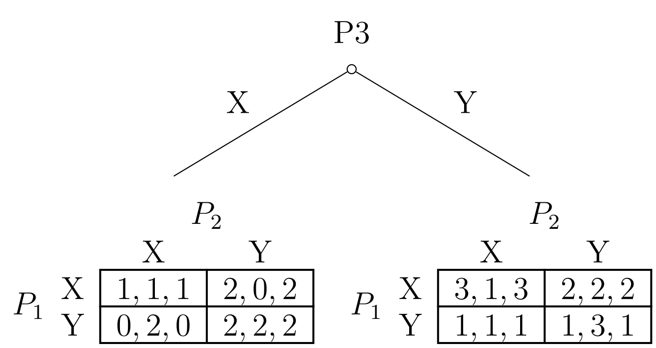
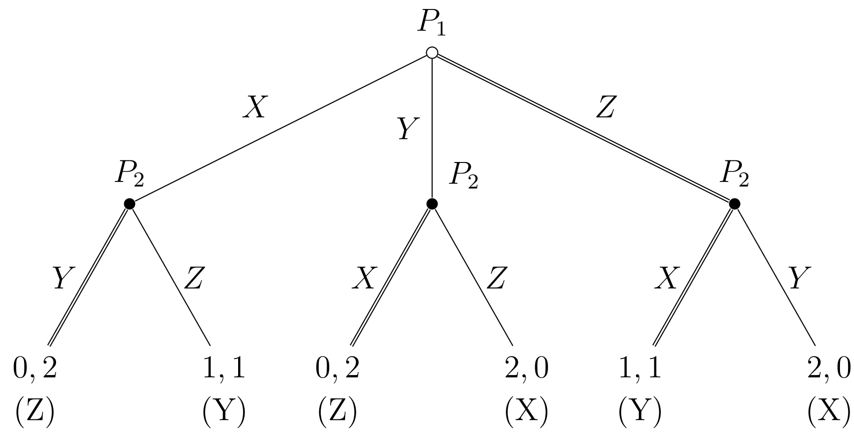


Figure 17: Veto Game in Extensive-Form (One Node for P_1 and Three Nodes for P_2)



Based on exercise 163.2 of Osborne's book:
Osborne, Martin J. 2004. *An Introduction to Game Theory.* Oxford: Oxford University Press.

Figure 18: Extensive-Form Game with Imperfect Information (Highlighting a Subgame)

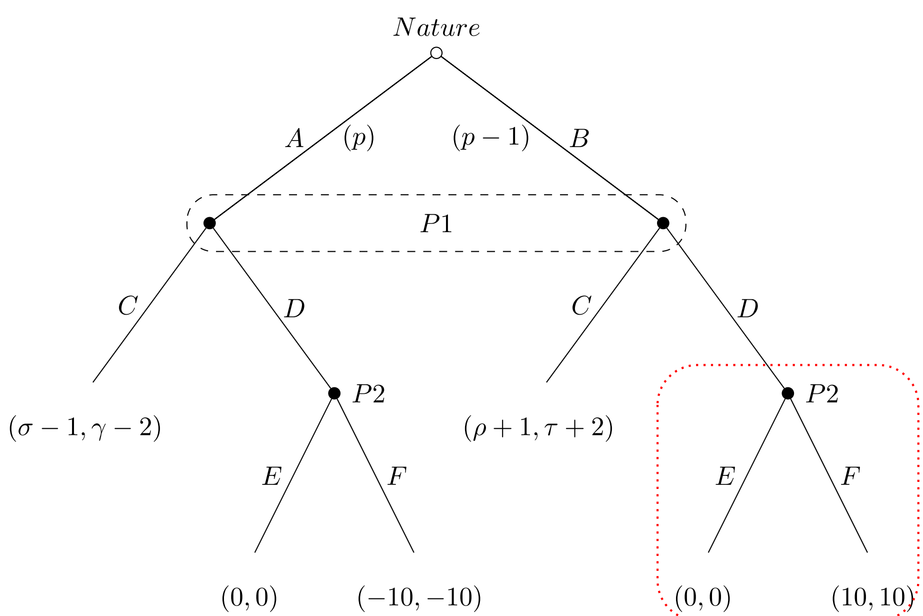
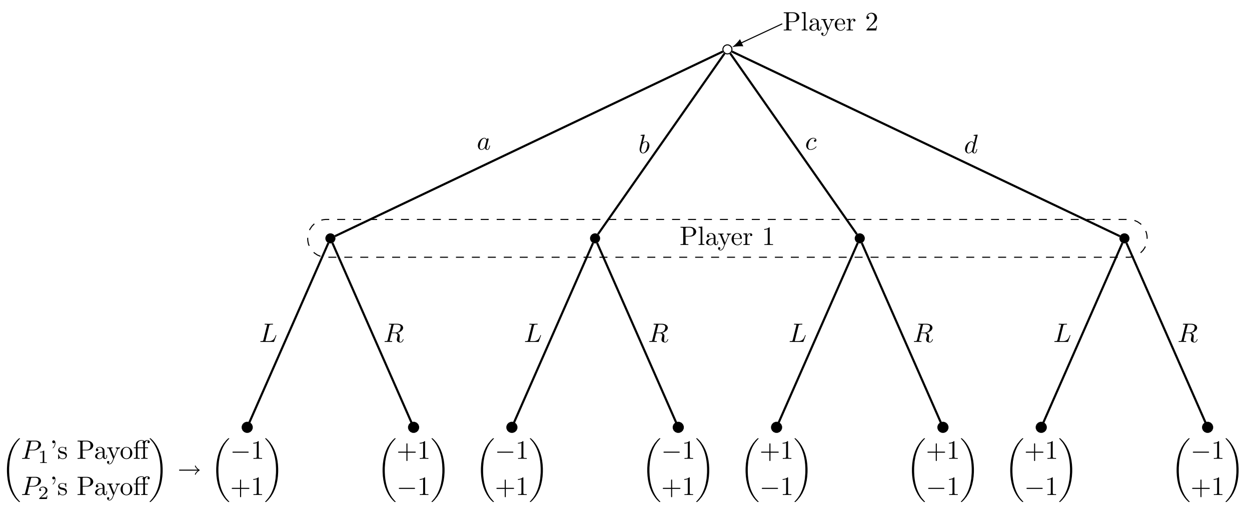
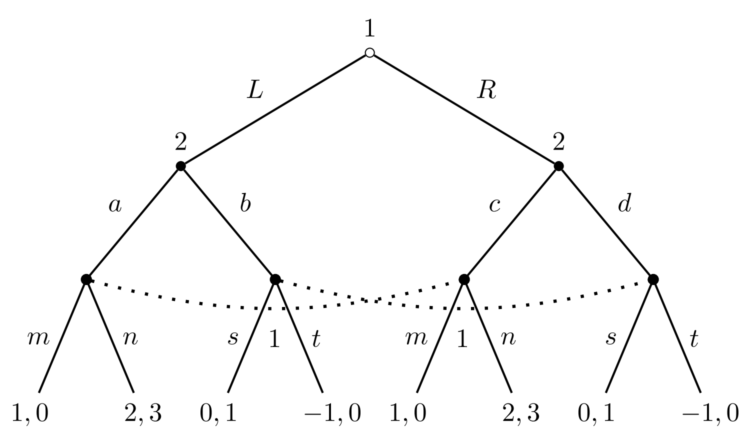


Figure 19: Extensive-Form Game with Imperfect Information and a Large Information Set



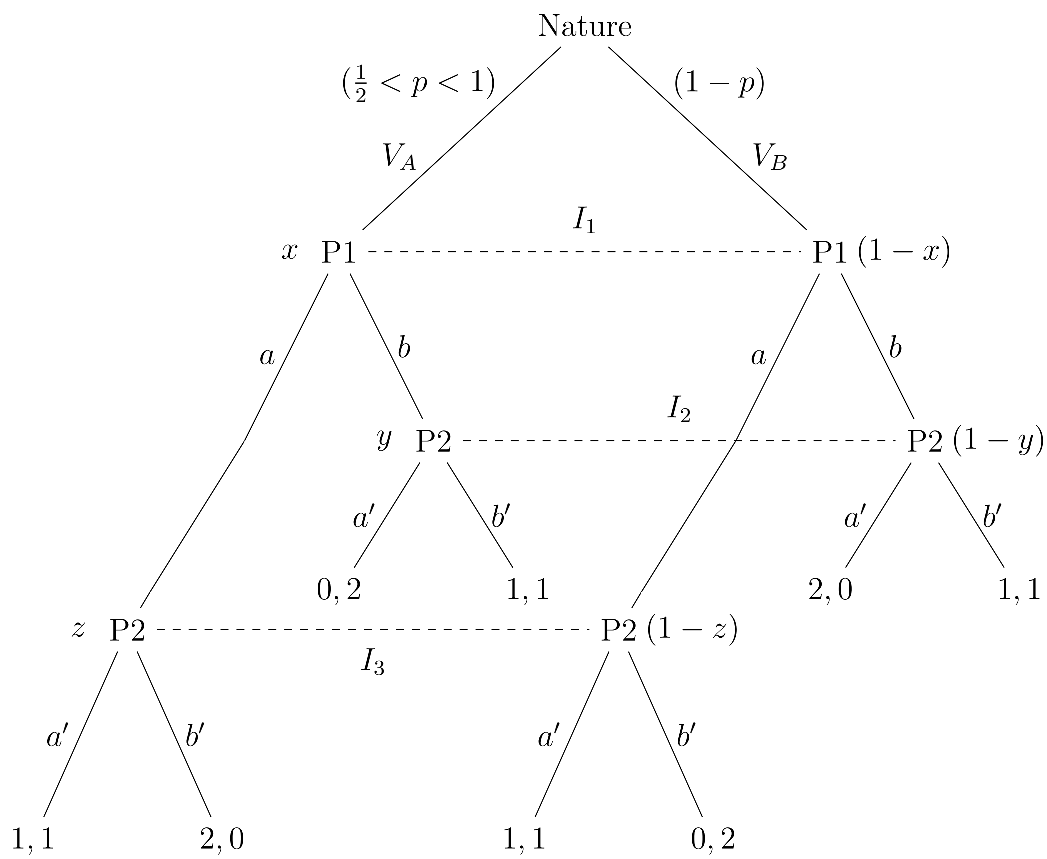
Note: This is Figure 7.D.2 in Mas-Colell, Whinston, and Green's book on microeconomic theory (1995), as replicated by Haiyun Chen (Simon Fraser University).

Figure 20: Game Tree with Curved Information Set



Note: This is Figure 6 in Osborne's "Manual for egameps.sty."

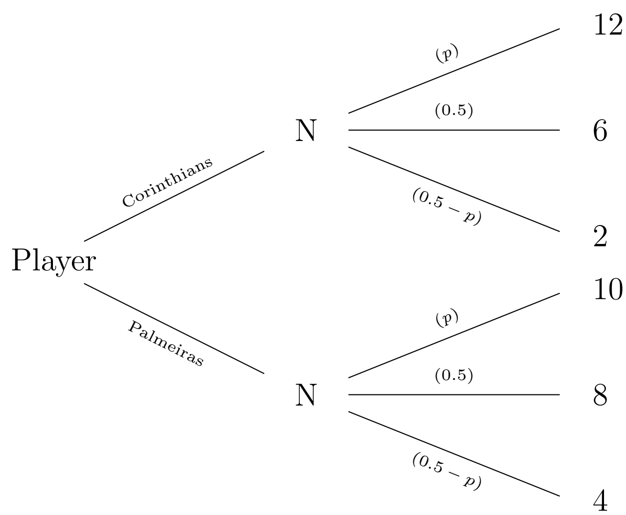
Figure 21: Long Game Tree with Three Information Sets



Note: This game tree was drew by Austin Mitchell (Texas A&M University).

We can also depict a game drawing a horizontal decision tree (horizontal extensive-form):

Figure 22: Game in Horizontal Extensive-Form



Expected utilities (EU) of the soccer player based on Figure 22:

$$v(\text{Corinthians}) = (p)(12) + (\tfrac{1}{2})(6) + (\tfrac{1}{2} - p)(2)$$

$$= 12p + 3 + 1 - 2p = 10p + 4$$

and

$$v(\text{Palmeiras}) = (p)(10) + (\tfrac{1}{2})(8) + (\tfrac{1}{2} - p)(4)$$

$$= 10p + 4 + 2 - 4p = 4p + 6$$

So, the soccer player should play in Corinthians if :

$$10p + 4 > 4p + 6 \quad (-4p)$$

$$6p + 4 > 6 \quad (-4)$$

$$6p > 2 \quad (\div 6)$$

$$p > \tfrac{2}{6} \quad (\div 2)$$

$$p > \tfrac{1}{3}$$

Figure 23: A More Complex Horizontal Decision Tree

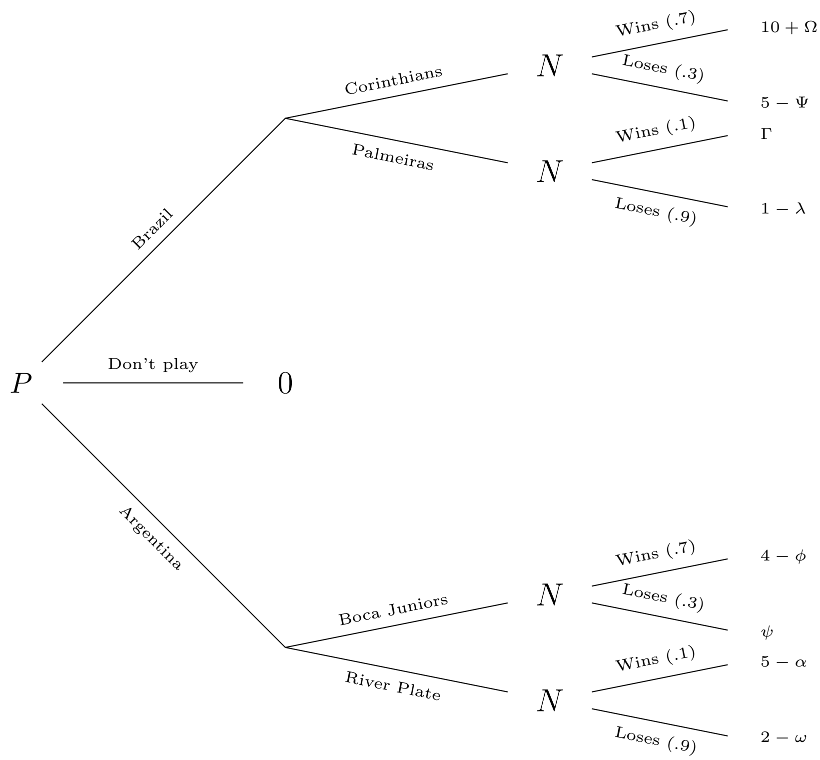
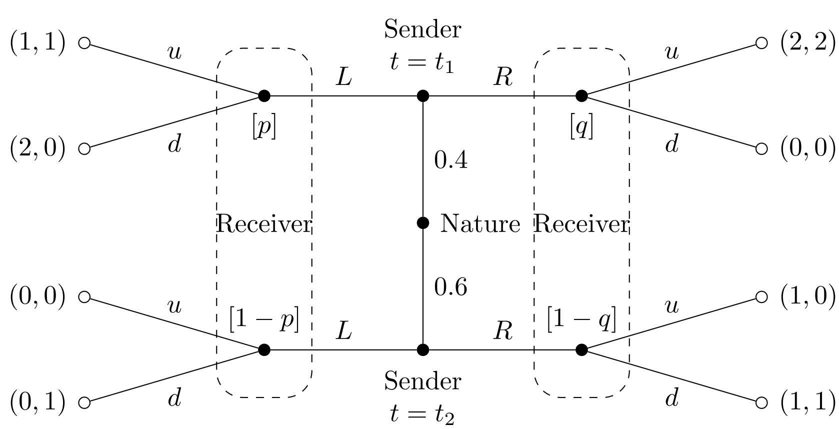


Figure 24: Signaling Game



Note: This game was drew by Chiu Yu Ko (National University of Singapore).

One-shot deviation principle (OSD)

$$\begin{array}{r|cc|cc|l} V1 & C & \text{nr} & C & \text{nr} & \ldots \\ \text{stick} \leftarrow V2 & C & \text{r} & C & \text{nr} & \ldots \quad (C-K)+\frac{\delta R}{1-\delta} = (C-K)+\delta R+\frac{\delta^2 R}{1-\delta} \end{array}$$

$$\begin{array}{r|cc|cc|cc|l} V1 & C & \text{nr} & D & \text{nr} & C & \text{nr} & \ldots \\ \text{OSD} \leftarrow V2 & C & \text{nr} & D & \text{r} & C & \text{nr} & \ldots \quad C+\delta(P-K)+\frac{\delta^2 R}{1-\delta} \end{array}$$

Sticking is optimal if,

$$(C-K)+\delta R \geq C+\delta(P-K)$$

$$\delta(K+R-P) \geq K$$

$$\boxed{\delta \geq \frac{K}{K+R-P}} \in (0,1)$$